平成３１年度第１次募集（平成３０年１０月入学含む）
新潟大学大学院自然科学研究科博士前期課程入学者選抜試験問題
外国人留学生特別入試

数理物質科学専攻

数理科学

A3

専門科目（数学）

注意事項

1. この問題冊子は，試験開始の合図があるまで開いてはいけません。
2. 問題冊子は，表紙を含めて全部で７ページあります。
3. 試験時間は　９：００～１１：００　です。
4. 試験開始後，次のものが配布されているか確認してください。

   問題冊子１部，解答用紙３枚

5. 問題は全部で６題あります。そのうち３題を選択して解答してください。
6. 各解答用紙には，問題番号と受験番号を記入してください。解答しない場合でも提出してください。
7. 下書きは，問題冊子の余白を使用してください。
8. 試験終了後，問題冊子は各自持ち帰ってください。

Problem 1

The gamma function $\Gamma(x)$ is defined for $x > 0$ as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

Also, the beta function $B(x, y)$ is defined as

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$

with $x > 0$ and $y > 0$. Then, answer the following questions.

(1) Calculate $\Gamma(n)$ for any natural number n.

(2) Show $B(x, y) = B(y, x)$.

(3) Prove that the gamma and beta functions are related to each other through

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)B(x, y).$$

(4) Calculate the definite integral given by

$$I = \int_{-1}^{1} \left(1 - t^2\right)^n dt,$$

where n is any natural number.

Problem 2

For the matrix $A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & 3 & -2 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, answer the following questions.

(1) Find the eigenvalues α, β, γ of A where α has algebraic multiplicity 2.

(2) Find a basis of each eigenspace of A.

(3) Find a matrix P such that $P^{-1}AP = \begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}$.

Problem 3

Let $\mathbb{C}$ be the set of all complex numbers and let $\mathrm{M}_3(\mathbb{C})$ be the set of all 3×3 matrices whose entries are complex numbers. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

as a linear operator on $\mathbb{C}^3$ by the matrix operation. We also denote by $\langle \xi, \eta \rangle$ the inner product of $\xi, \eta \in \mathbb{C}^3$. When $T \in \mathrm{M}_3(\mathbb{C})$, $T \geq 0$ means that

$$\langle T\xi, \xi \rangle \geq 0$$

for any $\xi \in \mathbb{C}^3$, and T^* is the conjugate transposed matrix of T. We define $\ker T$ as

$$\ker T = \{\xi \in \mathbb{C}^3 | \, T\xi = 0\}.$$

Then, answer the following questions.

(1) Find $P \in \mathrm{M}_3(\mathbb{C})$ which satisfies the conditions $P \geq 0$ and $P^2 = A^*A$.

(2) Find $U \in \mathrm{M}_3(\mathbb{C})$ which satisfies the conditions $A = UP$, $UU^*U = U$ and $\ker U = \ker A$.

Problem 4

Let $\mathbb{C}$ be the set of all complex numbers, and put $\mathbb{D} = \{a \in \mathbb{C} \mid |a| < 1\}$. For any $a, b \in \mathbb{D}$, define $a \oplus b$ as

$$a \oplus b = \frac{a+b}{1+\overline{a}b}$$

by using arithmetic and conjugate operations of complex numbers. Then, answer the following questions.

(1) Show $a \oplus b \in \mathbb{D}$.

(2) For $a = \frac{1}{2}$, $b = \frac{i}{2}$, $c = \frac{2+2i}{-4+i}$, find the values $a \oplus (b \oplus c)$ and $(a \oplus b) \oplus c$, respectively.

(3) For any $a, b \in \mathbb{D}$, express $z \in \mathbb{C}$ as a function of two variables a and b satisfying the condition

$$a \oplus b = z(b \oplus a).$$

(4) For any $a, b, c \in \mathbb{D}$, show the identity

$$a \oplus (b \oplus c) = (a \oplus b) \oplus (zc),$$

where z is given in (3).

Problem 5

Let Q_n $(n \geq 1)$ denote an n-dimentional hypercube graph with

$$V(Q_n) = \{(a_1, \ldots, a_n) | a_i \in \{0, 1\}\} \text{ and } E(Q_n) = \{xy | d_h(x, y) = 1, \quad x, y \in V(Q_n)\},$$

where $d_h(x, y) = |\{i | a_i \neq b_i\}|$ for two vertices $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$. Here, $|S|$ is the number of elements of a set S. Then, answer the following questions.

(1) Evaluate the number of vertices and the number of edges of Q_n, respectively.

(2) Prove that the distance between two vertices x and y of Q_n equals $d_h(x, y)$. Furthermore, evaluate the diameter of Q_n.

(3) Prove that Q_n contains no odd cycle.

Problem 6

A random variable Y is related to a normal random variable X with mean μ and variance σ^2 through $Y = e^X$. Then, answer the following questions.

(1) Obtain the probability density function $f_Y(y)$ of Y. Note that the probability density function $f_X(x)$ of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

(2) Calculate the mean of Y.

(3) Calculate the median of Y.

(4) Calculate the mode of Y.